A first-order conservation law framework for large strain compressible solids: isothermal viscoelasticity

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Many viscoelastic constitutive models [1] have been extensively applied in practice. However, under large strain regimes, these models may exhibit material instability due to the loss of ellipticity [2], [3]. This instability compromises the hyperbolicity of the solid dynamics formulation, leading to pressure and shear wave speeds that may not remain strictly positive for all deformation states. In explicit numerical schemes, wave speed computation is crucial for two reasons. First, accurate wave speed evaluation enhances the performance of Riemann solvers, improving resolution across jump interfaces. Second, wave speeds impact the time step size. Despite advances in numerical discretisation techniques, many Smoothed Particle Hydrodynamics (SPH) formulations rely on fictitious (or linearised) wave speeds, which often require problem-specific tuning for robustness. These challenges highlight the importance of ensuring the material stability of viscoelastic models before designing suitable numerical schemes. *Ad-hoc* regularisation techniques (i.e., Laplacian viscosity) are often implemented to maintain algorithmic stability but can compromise the physical accuracy of the model.

The aim of the current work is to develop well-posed viscoelastic models, followed by the formulation of suitable SPH schemes to achieve both numerical robustness and accuracy. Motivated by these considerations, we present a computational framework for large-strain solid dynamics in compressible viscoelasticity. The methodology is based on a system of first-order conservation laws [4]–[6], expressed in terms of the linear momentum and a triplet of deformation measures (e.g., the deformation gradient tensor, its co-factor and its Jacobian). To ensure both hyperbolicity and the existence of a convex *Hamiltonian* for symmetrisation [7], [8], the polyconvex Mooney-Rivlin model is adopted. This work extends the polyconvex hyperelastic model introduced in [4]–[6] to isothermal viscoelasticity [1], offering several key contributions. First, a well-posed compressible Mooney-Rivlin-type viscoelastic model, coupled with the linear evolution of internal state variables, is proposed. This formulation ensures material stability from a continuum perspective by expressing the strain energy density as a convex combination of the deformation measures. Second, the eigenvalue structure of the hyperbolic system is analysed to derive optimum time-step bounds for explicit time integrators. Third, numerical stability in the SPH scheme is enhanced through the introduction of consistently derived Godunov-type upwind numerical dissipation. A new approach is presented to demonstrate both numerical stabilisation and physical entropy production (i.e., Maxwell-type viscosity) in terms of the time rate of the *Hamiltonian* energy. Finally, the applicability and robustness of the proposed framework are validated through a series of numerical examples, where the SPH scheme is benchmarked against an in-house finite volume implementation.

I. Conservation laws for solid dynamics

Let us consider the motion of a continuum which in its material configuration is defined by a domain Ω_R of boundary $\partial\Omega_R$ with unit outward normal vector $\boldsymbol{N}$. After the motion, the continuum occupies a spatial configuration defined by a domain $\Omega(t)$ of boundary $\partial\Omega(t)$ with outward unit normal $\boldsymbol{n}$. The motion is described by a time-dependent mapping field $\boldsymbol{\varphi}(\boldsymbol{X},t)$ which links a material particle from material configuration $\boldsymbol{X}$ to spatial configuration $\boldsymbol{x}$ according to $\boldsymbol{x}=\boldsymbol{\varphi}(\boldsymbol{X},t)$. The motion can be described by a system of first-order conservation laws expressed in a Total Lagrangian setting as follows [4]–[6], [9], [11]–[14]

$$\frac{\partial \boldsymbol{p}_R}{\partial t} - \text{DIV}\boldsymbol{P} = \boldsymbol{f}_R; \tag{1a}$$
$$\frac{\partial \boldsymbol{F}}{\partial t} - \text{DIV}\left(\boldsymbol{v}\otimes\boldsymbol{I}\right) = \boldsymbol{0}; \tag{1b}$$
$$\frac{\partial \boldsymbol{H}}{\partial t} - \text{CURL}\left(\boldsymbol{v}\times\boldsymbol{F}\right) = \boldsymbol{0}; \tag{1c}$$
$$\frac{\partial J}{\partial t} - \text{DIV}\left(\boldsymbol{H}^T\boldsymbol{v}\right) = 0. \tag{1d}$$

Here, $\boldsymbol{p}_R=\rho_R\boldsymbol{v}$ is the linear momentum per unit of material volume, $\boldsymbol{v}$ is the velocity field, $\boldsymbol{F}$ is the deformation gradient (or fibre map), $\boldsymbol{H}$ is the co-factor of the deformation (or area map), J is the Jacobian of the deformation (or volume map), $\boldsymbol{P}$ is the first Piola-Kirchhoff stress tensor, and $\boldsymbol{f}_R$ is a body force term per unit of material volume. The symbol $\times$ represents the tensor cross product [4]–[6] between vectors and/or second order tensors, DIV and CURL represent the material divergence and material curl operators, respectively. Appropriate involutions [17] must be satisfied by the above system as

$$\text{CURL}\boldsymbol{F}=\boldsymbol{0};\quad \text{DIV}\boldsymbol{H}=\boldsymbol{0}. \tag{2}$$

In addition to initial and boundary (essential and natural) conditions required for the complete definition of the initial boundary value problem, closure of system (1a)-(1d) requires the introduction of a suitable constitutive law. This law must fulfill a series of physical principles, including thermodynamic consistency (via the Coleman–Noll procedure) and the principle of material frame indifference.

II. Extension to polyconvex viscoelasticity

Motivated by considerations of material stability [3], [4], [9], the concept of polyconvexity [2] is extended from isothermal hyperelasticity to isothermal viscoelasticity. The standard strain energy function is reformulated as a convex multi-variable function expressed as

$$W_R(\boldsymbol{\mathcal{X}}_{\boldsymbol{\mathcal{A}}}) = W_R^{\infty}(\boldsymbol{\mathcal{X}}) + \sum_{\alpha=1}^{n_{\text{Maxw}}} W_R^{\alpha}(\boldsymbol{\mathcal{X}}_{\boldsymbol{\mathcal{A}}}); \quad \boldsymbol{\mathcal{X}}_{\boldsymbol{\mathcal{A}}} = \{\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{A}}_\alpha\}. \tag{3}$$

Here, $\boldsymbol{\mathcal{X}} = \{\boldsymbol{F}, \boldsymbol{H}, J\}$ and $\boldsymbol{\mathcal{A}}_\alpha$ denotes a set of internal state variables for each Maxwell viscoelastic branch α.

For the long-term equilibrium component W_R^{∞}, we consider a polyconvex model given by a Mooney-Rivlin material as [15]

$$\begin{aligned} W_R^{\infty}(\boldsymbol{\mathcal{X}}) &= \xi_R(\boldsymbol{F}:\boldsymbol{F}-3) + \zeta_R(\boldsymbol{H}:\boldsymbol{H}-3) + f_R(J); \\ f_R(J) &= -2(\xi_R + 2\zeta_R)\ln J + \frac{\hat{\lambda}_R}{2}(J-1)^2. \end{aligned} \tag{4}$$

The parameters ξ_R, ζ_R and $\hat{\lambda}_R$ are positive, and they can be related through the relationships $\xi_R + \zeta_R = \frac{\mu_R}{2}$ and $\lambda_R = \hat{\lambda}_R + 4\zeta_R$, where μ_R and λ_R represent the shear modulus and Lamé parameter of the material.

Since the long-term strain energy W_R^{∞} (4) considered is isotropic function, the non-equilibrium viscous contributions W_R^{α} can be scaled to the shear-related terms of the Mooney-Rivlin model (4). Furthermore, the Mooney-Rivlin type viscous contributions are now extended to include both $\boldsymbol{C}_{v_\alpha}^{-1}$ and $\boldsymbol{G}_{v_\alpha}^{-1}$. Consequently, the internal state variables for each Maxwell branch defined as $\boldsymbol{\mathcal{A}}_\alpha = \{\boldsymbol{C}_{v_\alpha}^{-1}, \boldsymbol{G}_{v_\alpha}^{-1}\}$ is considered. With this extension, W_R^{α} is decomposed into two components as follows

$$W_R^{\alpha}(\boldsymbol{\mathcal{X}}_{\boldsymbol{\mathcal{A}}}) = \mathcal{C}_R^{\alpha}\left(\boldsymbol{F}, J, \boldsymbol{C}_{v\alpha}^{-1}\right) + \mathcal{G}_R^{\alpha}\left(\boldsymbol{H}, J, \boldsymbol{G}_{v\alpha}^{-1}\right). \tag{5}$$

The first component, $\mathcal{C}_R^{\alpha}\left(\boldsymbol{F}, J, \boldsymbol{C}_{v\alpha}^{-1}\right)$, represents the viscous effects arising from deformation changes associated with the fibre mapping, characterised by $\boldsymbol{C}_{v_\alpha}^{-1}$. The second component, $\mathcal{G}_R^{\alpha}\left(\boldsymbol{H}, J, \boldsymbol{G}_{v\alpha}^{-1}\right)$, captures the viscous effects linked to the inverse of $\boldsymbol{G}_{v_\alpha}$, which describe the material response through area mapping. Their corresponding expressions are given by

$$\begin{aligned} \mathcal{C}_R^{\alpha} &= \xi_R^{\alpha}\left(\boldsymbol{C}:\boldsymbol{C}_{v_\alpha}^{-1} - 3\right) - 2\xi_R^{\alpha}\ln J - \xi_R^{\alpha}\ln\left(\det \boldsymbol{C}_{v_\alpha}^{-1}\right); \\ \mathcal{G}_R^{\alpha} &= \zeta_R^{\alpha}\left(\boldsymbol{G}:\boldsymbol{G}_{v_\alpha}^{-1} - 3\right) - 4\zeta_R^{\alpha}\ln J - \zeta_R^{\alpha}\ln\left(\det \boldsymbol{G}_{v_\alpha}^{-1}\right). \end{aligned} \tag{6}$$

Here, $\xi_R^{\alpha} = \beta_{\boldsymbol{C}_{v_\alpha}}\xi_R$ and $\zeta_R^{\alpha} = \beta_{\boldsymbol{G}_{v_\alpha}}\zeta_R$, where $\beta_{\boldsymbol{C}_{v_\alpha}}$ and $\beta_{\boldsymbol{G}_{v_\alpha}}$ are positive non-dimensional proportionality factors. The viscous contribution for a neo-Hookean model can be recovered by setting $\zeta_R = 0$, which eliminates $\mathcal{G}_R^{\alpha}$.

Once both the long-term and viscous strain energy functions are defined, the next step is to derive the first Piola-Kirchhoff stress tensor. To achieve this, we first introduce the conjugate stresses with respect to the deformation measure triplet $\{\boldsymbol{F}, \boldsymbol{H}, J\}$ as

$$\boldsymbol{\Sigma}_{\boldsymbol{F}} = \frac{\partial W_R(\boldsymbol{\mathcal{X}}_{\boldsymbol{\mathcal{A}}})}{\partial \boldsymbol{F}}; \quad \boldsymbol{\Sigma}_{\boldsymbol{H}} = \frac{\partial W_R(\boldsymbol{\mathcal{X}}_{\boldsymbol{\mathcal{A}}})}{\partial \boldsymbol{H}}; \quad \Sigma_J = \frac{\partial W_R(\boldsymbol{\mathcal{X}}_{\boldsymbol{\mathcal{A}}})}{\partial J}. \tag{7}$$

Differentiating the convex strain energy (3) with respect to time, while keeping the set of internal state variables $\boldsymbol{\mathcal{A}}_\alpha$ constant, results in

$$\begin{aligned} \boldsymbol{P}:\dot{\boldsymbol{F}} &= \left.\frac{dW_R}{dt}\right|_{\boldsymbol{\mathcal{A}}_\alpha=\text{const}} \\ &= \frac{\partial W_R(\boldsymbol{\mathcal{X}}_{\boldsymbol{\mathcal{A}}})}{\partial \boldsymbol{F}}:\dot{\boldsymbol{F}} + \frac{\partial W_R(\boldsymbol{\mathcal{X}}_{\boldsymbol{\mathcal{A}}})}{\partial \boldsymbol{H}}:\dot{\boldsymbol{H}} + \frac{\partial W_R(\boldsymbol{\mathcal{X}}_{\boldsymbol{\mathcal{A}}})}{\partial J}\dot{J} \\ &= (\boldsymbol{\Sigma}_{\boldsymbol{F}} + \boldsymbol{\Sigma}_{\boldsymbol{H}} \times \boldsymbol{F} + \Sigma_J \boldsymbol{H}):\dot{\boldsymbol{F}}. \end{aligned} \tag{8}$$

Here, the expressions $\dot{\boldsymbol{H}} = \boldsymbol{F} \times \dot{\boldsymbol{F}}$ and $\dot{J} = \boldsymbol{H}:\dot{\boldsymbol{F}}$ are substituted in the second line. Comparing the expression above yields the following relation

$$\boldsymbol{P} = \boldsymbol{\Sigma}_{\boldsymbol{F}} + \boldsymbol{\Sigma}_{\boldsymbol{H}} \times \boldsymbol{F} + \Sigma_J \boldsymbol{H}. \tag{9}$$

Utilising expression (3), each conjugate stress comprises two components, namely a long-term equilibrium part and a set of viscous contribution. These are expressed as

$$\boldsymbol{\Sigma}_{\boldsymbol{F}}(\boldsymbol{\mathcal{X}}_{\boldsymbol{\mathcal{A}}}) = \boldsymbol{\Sigma}_{\boldsymbol{F}}^{\infty}(\boldsymbol{\mathcal{X}}) + \sum_{\alpha=1}^{n_{\text{Maxw}}} \boldsymbol{\Sigma}_{\boldsymbol{F}}^{\alpha}(\boldsymbol{\mathcal{X}}_{\boldsymbol{\mathcal{A}}}); \tag{10a}$$

$$\boldsymbol{\Sigma}_{\boldsymbol{H}}(\boldsymbol{\mathcal{X}}_{\boldsymbol{\mathcal{A}}}) = \boldsymbol{\Sigma}_{\boldsymbol{H}}^{\infty}(\boldsymbol{\mathcal{X}}) + \sum_{\alpha=1}^{n_{\text{Maxw}}} \boldsymbol{\Sigma}_{\boldsymbol{H}}^{\alpha}(\boldsymbol{\mathcal{X}}_{\boldsymbol{\mathcal{A}}}); \tag{10b}$$

$$\Sigma_J(\boldsymbol{\mathcal{X}}_{\boldsymbol{\mathcal{A}}}) = \Sigma_J^{\infty}(\boldsymbol{\mathcal{X}}) + \sum_{\alpha=1}^{n_{\text{Maxw}}} \Sigma_J^{\alpha}(\boldsymbol{\mathcal{X}}_{\boldsymbol{\mathcal{A}}}). \tag{10c}$$

As an illustration, considering the Mooney-Rivlin model described in (4) and (5), the long-term $\{\boldsymbol{\Sigma}_{\boldsymbol{F}}^{\infty}, \boldsymbol{\Sigma}_{\boldsymbol{H}}^{\infty}, \Sigma_J^{\infty}\}$ and non-equilibrium components $\{\boldsymbol{\Sigma}_{\boldsymbol{F}}^{\alpha}, \boldsymbol{\Sigma}_{\boldsymbol{H}}^{\alpha}, \Sigma_J^{\alpha}\}$ of the conjugate stresses are given by

$$\boldsymbol{\Sigma}_{\boldsymbol{F}}^{\infty}(\boldsymbol{\mathcal{X}}) = 2\xi_R \boldsymbol{F}; \qquad \boldsymbol{\Sigma}_{\boldsymbol{F}}^{\alpha}(\boldsymbol{\mathcal{X}}_{\boldsymbol{\mathcal{A}}}) = 2\xi_R^{\alpha}\boldsymbol{F}\boldsymbol{C}_{v_\alpha}^{-1}; \tag{11a}$$

$$\boldsymbol{\Sigma}_{\boldsymbol{H}}^{\infty}(\boldsymbol{\mathcal{X}}) = 2\zeta_R \boldsymbol{H}; \qquad \boldsymbol{\Sigma}_{\boldsymbol{H}}^{\alpha}(\boldsymbol{\mathcal{X}}_{\boldsymbol{\mathcal{A}}}) = 2\zeta_R^{\alpha}\boldsymbol{H}\boldsymbol{G}_{v_\alpha}^{-1}; \tag{11b}$$

$$\Sigma_J^{\infty}(\boldsymbol{\mathcal{X}}) = -\frac{2(\xi_R + 2\zeta_R)}{J} + \hat{\lambda}_R(J-1); \quad \Sigma_J^{\alpha} = -\frac{2(\xi_R^{\alpha} + 2\zeta_R^{\alpha})}{J}. \tag{11c}$$

To complete the description of the proposed viscoelastic model, it is necessary to define the evolution equations for the internal variables represented by the viscous strain tensors, namely $\boldsymbol{C}_{v_\alpha}^{-1}$ and $\boldsymbol{G}_{v_\alpha}^{-1}$. This can be achieved by recalling the dissipation inequality, which accounts for the time derivative of the strain energy while keeping the deformation measures $\boldsymbol{\mathcal{X}}$ constant, as described by

$$0 \le \mathcal{D}_{\text{int}}(\boldsymbol{\mathcal{X}}_{\boldsymbol{\mathcal{A}}}) = -\sum_{\alpha=1}^{n_{\text{Maxw}}}\left[\boldsymbol{\Sigma}_{\boldsymbol{C}_{v_\alpha}^{-1}}:\dot{\boldsymbol{C}}_{v_\alpha}^{-1} + \boldsymbol{\Sigma}_{\boldsymbol{G}_{v_\alpha}^{-1}}:\dot{\boldsymbol{G}}_{v_\alpha}^{-1}\right]. \tag{12}$$

The non-equilibrium conjugate stresses are defined as

$$\boldsymbol{\Sigma}_{\boldsymbol{C}_{v_\alpha}^{-1}} = \xi_R^{\alpha}\left(\boldsymbol{C} - \boldsymbol{C}_{v_\alpha}^{-1}\right); \quad \boldsymbol{\Sigma}_{\boldsymbol{G}_{v_\alpha}^{-1}} = \zeta_R^{\alpha}\left(\boldsymbol{G} - \boldsymbol{G}_{v_\alpha}^{-1}\right). \tag{13}$$

In order to satisfy the dissipation inequality given in equation (12), the linear evolution equations for $\boldsymbol{C}_{v\alpha}^{-1}$ and $\boldsymbol{G}_{v\alpha}^{-1}$ can be obtained as follows [16]

$$\left.\frac{d\boldsymbol{C}_{v_\alpha}^{-1}}{dt}\right|_{\boldsymbol{\mathcal{X}}=\text{const}} = \frac{1}{\tau_{\boldsymbol{C}_{v_\alpha}^{-1}}}\left(\boldsymbol{C}^{-1}-\boldsymbol{C}_{v_\alpha}^{-1}\right); \tag{14a}$$

$$\left.\frac{d\boldsymbol{G}_{v_\alpha}^{-1}}{dt}\right|_{\boldsymbol{\mathcal{X}}=\text{const}} = \frac{1}{\tau_{\boldsymbol{G}_{v_\alpha}^{-1}}}\left(\boldsymbol{G}^{-1}-\boldsymbol{G}_{v_\alpha}^{-1}\right); \quad \alpha=\{1,\dots,n_{\text{Maxw}}\}. \tag{14b}$$

If the viscous strains equal to the total strains ($\boldsymbol{C}_{v_\alpha}^{-1}=\boldsymbol{C}^{-1}$ and $\boldsymbol{G}_{v_\alpha}^{-1}=\boldsymbol{G}^{-1}$), the internal variables remain constant, indicating that no viscosity is involved in the deformation process.

A. Combined equations

Combining the conservation of linear momentum (1a), the triplet set of geometric conservation equations (1b)-(1d), and the evolution equations for the internal state variables (14a) and (14b) for each Maxwell branch α, the governing system for a Mooney-Rivlin-type viscoelastic model accounting for isothermal processes can be expressed in hyperbolic form as

$$\frac{\partial \boldsymbol{\mathcal{U}}_R}{\partial t}+\sum_{I=1}^{3}\frac{\partial \boldsymbol{\mathcal{F}}_R^I}{\partial X_I}=\boldsymbol{\mathcal{S}}_R, \tag{15}$$

where $\boldsymbol{\mathcal{U}}_R$ denotes the set of conservation variables, $\boldsymbol{\mathcal{S}}_R$ represents the source term, and $\boldsymbol{\mathcal{F}}_R^I$ is the flux vector in the material Cartesian direction I, given by

$$\boldsymbol{\mathcal{U}}_R=\begin{bmatrix}\boldsymbol{p}_R\\ \boldsymbol{F}\\ \boldsymbol{H}\\ J\\ \boldsymbol{C}_{v_\alpha}^{-1}\\ \boldsymbol{G}_{v_\alpha}^{-1}\end{bmatrix}; \qquad \boldsymbol{\mathcal{F}}_R^I=-\begin{bmatrix}\boldsymbol{P}\boldsymbol{E}_I\\ \boldsymbol{v}\otimes\boldsymbol{E}_I\\ \boldsymbol{F}\times(\boldsymbol{v}\otimes\boldsymbol{E}_I)\\ \boldsymbol{H}:(\boldsymbol{v}\otimes\boldsymbol{E}_I)\\ \boldsymbol{0}\\ \boldsymbol{0}\end{bmatrix} \tag{16}$$

and

$$\boldsymbol{\mathcal{S}}_R=\begin{bmatrix}\boldsymbol{f}_R\\ \boldsymbol{0}\\ \boldsymbol{0}\\ 0\\ \frac{1}{\tau_{\boldsymbol{C}_{v_\alpha}^{-1}}}\left(\boldsymbol{C}^{-1}-\boldsymbol{C}_{v_\alpha}^{-1}\right)\\ \frac{1}{\tau_{\boldsymbol{G}_{v_\alpha}^{-1}}}\left(\boldsymbol{G}^{-1}-\boldsymbol{G}_{v_\alpha}^{-1}\right)\end{bmatrix}. \tag{17}$$

Here, $\boldsymbol{E}_I$ is the I-th unit vector of the Cartesian basis.

III. SPH SPATIAL DISCRETISATION

Following Reference [14], the resulting SPH formulations for $\{\boldsymbol{p}_R,\boldsymbol{F},\boldsymbol{H},J\}$ are expresssed as follows

$$\Omega_R^a\frac{d\boldsymbol{p}_R^a}{dt}=\sum_{b\in\Lambda_a^b}\frac{1}{2}\left(\boldsymbol{P}_a\boldsymbol{C}_{ab}-\boldsymbol{P}_b\boldsymbol{C}_{ba}\right)+A_R^a\boldsymbol{t}_B^a+\sum_{b\in\Lambda_a^b}\boldsymbol{\mathcal{D}}_{ab}; \tag{18a}$$

$$\Omega_R^a\frac{d\boldsymbol{F}_a}{dt}=\sum_{b\in\Lambda_a^b}\frac{1}{2}\left(\boldsymbol{v}_b-\boldsymbol{v}_a\right)\otimes\boldsymbol{C}_{ab}; \tag{18b}$$

$$\Omega_R^a\frac{d\boldsymbol{H}_a}{dt}=\boldsymbol{F}_a\times\left(\sum_{b\in\Lambda_a^b}\frac{1}{2}\left(\boldsymbol{v}_b-\boldsymbol{v}_a\right)\otimes\boldsymbol{C}_{ab}\right); \tag{18c}$$

$$\Omega_R^a\frac{dJ_a}{dt}=\boldsymbol{H}_a:\left(\sum_{b\in\Lambda_a^b}\frac{1}{2}\left(\boldsymbol{v}_b-\boldsymbol{v}_a\right)\otimes\boldsymbol{C}_{ab}\right). \tag{18d}$$

The pseudo-area vectors are given by [12]

$$\boldsymbol{C}_{ab}=2\Omega_R^a\Omega_R^b\tilde{\boldsymbol{\nabla}}_0W_b(\boldsymbol{X}_a); \qquad \boldsymbol{C}_{ba}=2\Omega_R^a\Omega_R^b\tilde{\boldsymbol{\nabla}}_0W_a(\boldsymbol{X}_b). \tag{19}$$

Additionally, the pairwise stabilisation terms $\boldsymbol{\mathcal{D}}_{ab}$ are derived using the semi-discrete form of the Coleman–Noll procedure. Specifically, the dissipation depends on velocity jumps, a characteristic feature of Riemann solvers [18].

Incorporating viscoelasticity within the first-order framework requires additional evolution equations for the associated set of internal state variables. The corresponding semi-discrete equations for particle a are given by

$$\frac{d\boldsymbol{C}_{v_\alpha,a}^{-1}}{dt}=\frac{1}{\tau_{\boldsymbol{C}_{v_\alpha}^{-1}}}\left(\boldsymbol{C}_a^{-1}-\boldsymbol{C}_{v_\alpha,a}^{-1}\right); \quad \frac{d\boldsymbol{G}_{v_\alpha,a}^{-1}}{dt}=\frac{1}{\tau_{\boldsymbol{G}_{v_\alpha}^{-1}}}\left(\boldsymbol{G}_a^{-1}-\boldsymbol{G}_{v_\alpha,a}^{-1}\right). \tag{20}$$

IV. NUMERICAL EXAMPLES

Several numerical examples will be presented and discussed during the conference, comparing the SPH results with those obtained using a vertex-centred finite volume algorithm [19].

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