A Coefficient-Free β-Velocity Formulation for Improved Riemann-Based SPH

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I. INTRODUCTION

Smoothed Particle Hydrodynamics (SPH) is a mesh-free numerical method widely used in computational fluid dynamics, particularly for scenarios involving complex geometries and free-surface flows. However, as a particle-based method, SPH faces challenges when dealing with flows that involve shocks and discontinuities. Monaghan [1] proposed an SPH formulation that closely resembling Riemann solutions in compressible gas dynamics. Subsequently, Pershikov and Medin [2] introduced a Riemann-based SPH formulation**,** where particle interactions are directly calculated using a Riemann solver**.** However, in their original form, Riemann-based SPH schemes are highly dissipative. Zhang et al. [3] proposed a β velocity formulation, which reduces the overall dissipation of the scheme and is widely used in Riemann-based SPH. However, this β velocity formulation is dependent on a tuning coefficient.

In this study, a comprehensive analysis is performed to evaluate the accuracy of the Riemann-based SPH method by introducing a novel β velocity formulation that requires no tuning coefficients. Additionally, a comparison is made between the newly proposed β velocity and that of [3], focusing on the accuracy of the scheme. Furthermore, MUSCL reconstruction is included in the numerical scheme to enhance its accuracy. The combined effects of the β velocity formulations and MUSCL reconstruction on the scheme's accuracy and stability are also thoroughly investigated. The detailed formulation of the numerical scheme is explained in the following section.

II. NUMERICAL SCHEME

Riemann problems are initial value problems with data that contain a discontinuity. In SPH, each particle interaction can be considered as a Riemann problem where each particle possesses distinct physical properties and interacts with neighbouring particles. In this study a weakly compressible SPH (WCSPH) model based on Riemann Solver developed by Rezavand et al. [4] is used. The intermediate state velocity (U^*) and pressure (P^*) of Riemann problem with identical speed of sound in both states can be calculated as

$$U^* = \frac{\rho_l U_l + \rho_r U_r}{\rho_l + \rho_r} + \frac{P_l - P_r}{c_o(\rho_l + \rho_r)}, \tag{1}$$

$$P^* = \frac{\rho_l P_r + \rho_r P_l}{\rho_l + \rho_r} + \beta \frac{\rho_l \rho_r (U_l - U_r)}{\rho_l + \rho_r}, \tag{2}$$

where ρ is the density, $\mathbf{v}$ is the velocity, p is the pressure, U is the velocity defined for Riemann problem and the subscripts l and r represents the left and right state of Riemann problem respectively. The continuity and momentum equations used by [4] are given as follows

$$\frac{d\rho_i}{dt} = 2\rho_i \sum_j \frac{m_j}{\rho_j} (\mathbf{v}_i - \mathbf{V}^*) \nabla_i W_{ij} \tag{3}$$

$$\frac{d\mathbf{v}_i}{dt} = -2 \sum_j m_j \frac{P^*}{\rho_i \rho_j} \nabla_i W_{ij} + \mathbf{f} \tag{4}$$

where d/dt is the Lagrangian derivative, m is the mass of the particle, W_{ij} is the smoothing kernel, i is considered as particle of interest, j represents the neighboring particles, $\mathbf{f}$ is the body force and $\mathbf{V}^*$ is given as $\mathbf{V}^* = U^* \mathbf{e}_{ij} + (\bar{\mathbf{v}} - \bar{U}. \mathbf{e}_{\mathbf{ij}})$. $\bar{\mathbf{v}}$ is the density-weighted average of the actual velocity of the particles with $\mathbf{e}$ the unit vector along which the Riemann problem is defined and $\bar{U}$ is the density weighted average of the left and right states velocity. The first term on the RHS of (2) represents the density-weighted averages, while the second term, which is the primary focus of this article, controls the diffusivity of formulation. Due to the assumption of weak compressibility, the Mach number of the flow is always small, and it is well known that Finite Volume (FV) Godunov schemes show low accuracy and excessive numerical dissipation for such flow regimes [5]. To reduce the overall numerical dissipation of the WCSPH scheme Zhang et al. [3] introduced a β velocity in the dissipation term defined as follows

$$\beta_{Zhang} = \min(\eta \max(U_l - U_r, 0), c_o), \tag{5}$$

where η is a numerical constant used to reduce the dissipation, independent of any physical quantity and according to [3] the best choice is $\eta = 3$. In the present work, we propose a new formulation for the β velocity based on local Mach number (Ma), inspired by the work of Chen et al. [5] in a Finite Volume scheme, which is independent of any tuning coefficient given as follows

$$\beta_{new} = c_o \begin{cases} min(1, Ma^2) & U_l < U_r \\ 0 & else \end{cases}, \qquad (6)$$

where,

$$Ma^2 = \min\left(\frac{|U_l|^2}{{c_o}^2}, \frac{|U_r|^2}{{c_o}^2}\right).$$

In this article alongside the two aforementioned formulations for β velocity we consider a case with $\beta_1 = c_0$ which corresponds to the standard formulation of the approximate Riemann solver herein adopted.

A. *MUSCL Reconstruction*

Monotonic Upstream-centered Scheme for Conservation Laws (MUSCL) is an effective and extensively employed technique for achieving both accuracy and stability while dealing with complex fluid dynamics problems that involve shocks, rarefactions, and other nonlinear phenomena. The updated velocity and pressure term obtained by using MUSCL reconstruction can be written as follows

$$\mathbf{v}_k = \mathbf{v}_\mathrm{k} + \frac{\varphi(r)}{2}\left(\mathbf{r}_{ij}.\frac{\partial \mathbf{v}_k}{\partial \mathbf{r}}\right),$$

$$P_k = P_k + \frac{\varphi(r)}{2}\left(\mathbf{r}_{ij}.\frac{\partial P_k}{\partial \mathbf{r}}\right),$$

where the subscript k can either be i or j depending on which particle we are dealing with. While $\partial/\partial\mathbf{r}$ represents the local derivative with respect to position vector and $\varphi(r)$ represent the slope limiters. In the present study, the Van Leer limiter is used, and the proposed techniques are implemented and evaluated using the open-source code DualSPHysics.

III. Test Cases

A. *Poiseuille Flow*

To analyse the impact of different formulations for the β velocity and MUSCL reconstruction on the accuracy of the scheme, the Poiseuille flow with $Re = 1000$ is used.

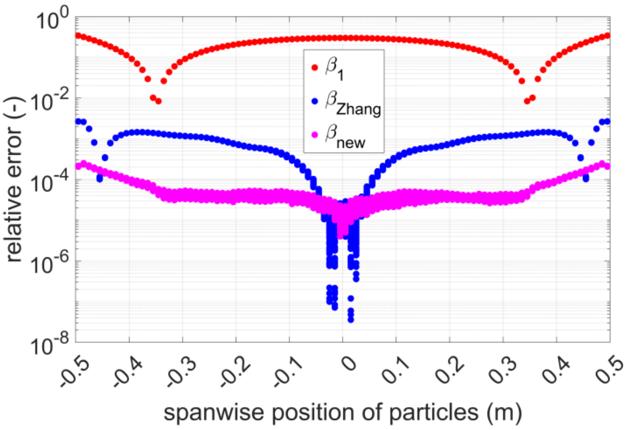

Figure 1. Relative horizontal velocity error for Poiseuille flow with Re=1000 using different formulations for the beta velocity

The steady-state analytical solution is adopted as the initial condition and $t = 1s$ of physical time is simulated. In Fig. 1, the relative horizontal velocity error for all particles is shown considering a particle spacing $dp = 0.01m$ and the smoothing length $h = 2dp$. It is evident that when using β_1, which corresponds to approximate Riemann solver, the scheme is over dissipative and has large errors, whereas, results have much smaller errors when (5) and (6) are adopted. The new proposed formulation for β velocity (β_{new}) is less dissipative and more accurate with maximum relative errors that are about one order of magnitude smaller than those obtained with the formulation proposed by Zhang et al. [3] (β_{Zhang}).

In Fig. 2, the maximum, minimum, and average values of the β velocity are plotted for all particle interactions. Clearly, β_{new} is proportional to the velocity profile, whereas β_{Zhang} is proportional to the velocity gradient, and this is the cause of much broader range of variation.

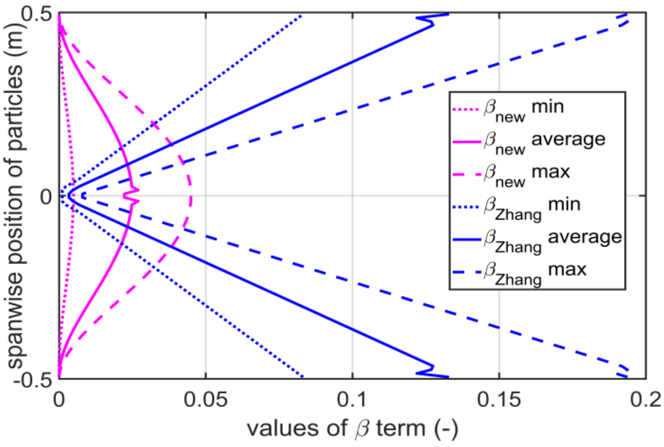

Figure 2. Minimum, maximum and average values of β_{new} and β_{Zhang} for Poiseuille flow with Re=1000

Fig. 3 shows the relative horizontal velocity error adopting the numerical scheme with MUSCL reconstruction which improves the accuracy of the formulations for all three β velocities with the formulations of Equations (5) and (6) still more accurate than the standard formulation. The L_2 norms of the horizontal velocity error without and with MUSCL reconstruction are shown in Fig. 4 and Fig. 5 respectively.

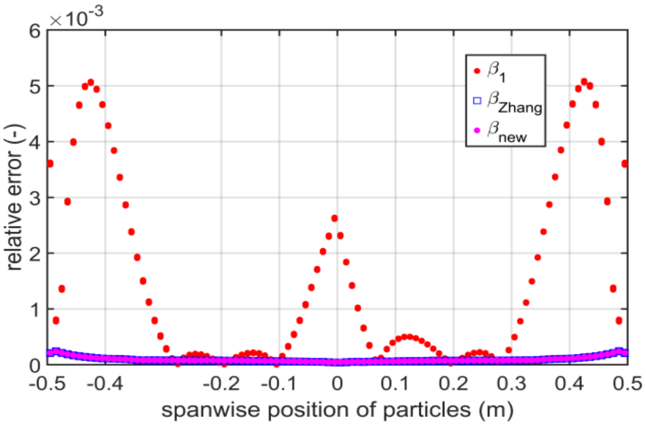

Figure 3. Relative error of horizontal component of velocity for Poiseuille flow with Re=1000 and MUSCL reconstruction

Without MUSCL reconstruction, β_1 $(\beta = c_0)$ exhibits a very low rate of convergence with an error norm which is significantly larger than the one obtained with other two formulations for all resolutions. Results obtained with the β_{new} are more accurate than those with β_{Zhang}.

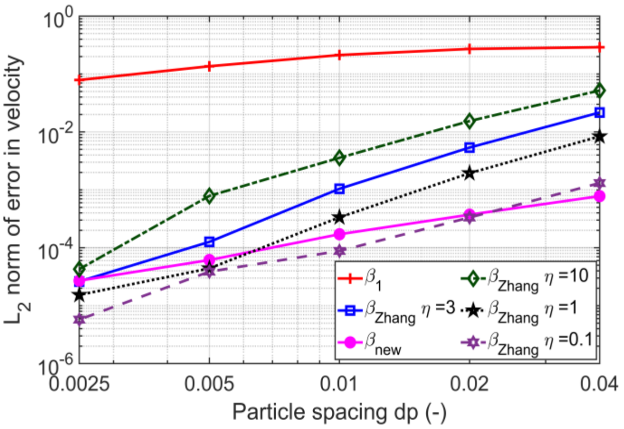

Figure 4. Convergence analysis for Poiseuille flow with Re=1000 without MUSCL reconstruction

The MUSCL Reconstruction can significantly increase the accuracy of the results if the β_1 velocity is adopted, β_{new} exhibits a significantly lower L_2 norm in comparison with the β_1 velocity, with very similar values to β_{Zhang}. Also, in Fig. 4 the convergence analysis for the β_{Zhang} formulation is shown considering the tuning coefficient η equals to 0.1, 1, 3 and 10. The L_2 norm of the velocity error is very sensitive to the tuning coefficient with best results obtained with $\eta = 0.1$. However for test cases which involve shock waves such as the patch test, values of the tuning coefficient $\eta < 1$ do not guarantee stability.

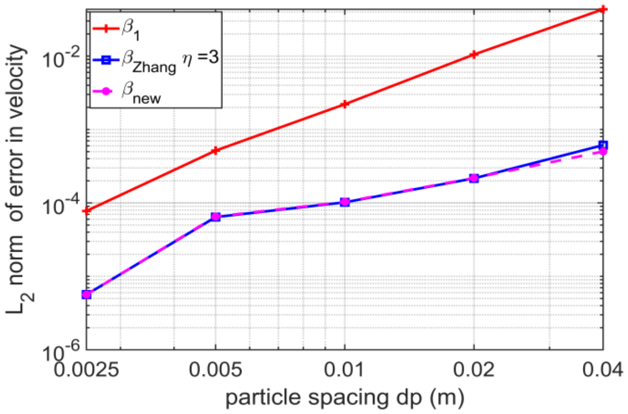

Figure 5. Convergence analysis for Poiseuille flow with Re=1000 with MUSCL reconstruction

B. Patch test case

The next test case considered is the impact of two identical inviscid rectangular jets moving at $1m/s$ toward each other. The length of each water jet is $1m$ and the width is $0.667m$. The impact causes a sudden loss of kinetic energy, and a shock wave that propagates through the domain and travels back and forth due to the assumption of weak compressibility with a speed depending on the adopted value of speed of sound. Fig. 6 and Fig. 7 show the non-dimensional pressure for all three β velocity formulations without and with MUSCL reconstruction at t=0.03s, with $dp = 0.005m$ and $h = 2dp$. Similar to the previous test case, β_1 is extremely dissipative, while the physics of the problem are reproduced with the other two β velocity formulations.

From Fig. 7 it is evident that MUSCL is highly effective in reducing dissipation. Due to space constraints, the additional results are not included here but will be presented and discussed at the conference.

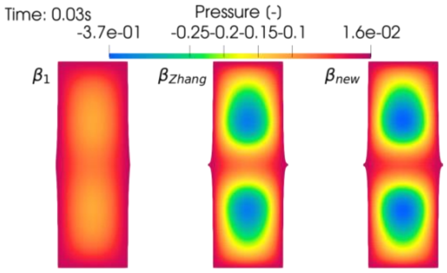

Figure 6. Pressure profile at t=0.03s without MUSCL Reconstruction

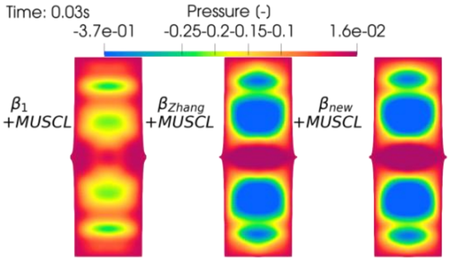

Figure 7. Pressure profile at t=0.03s with MUSCL Reconstruction

IV. Conclusion

In the present work we analysed different Riemann-based WCSPH method and proposed a new formulation for the β velocity. While the standard formulation is excessively dissipative and fails to achieve reasonable accuracy for the Poiseuille flow, the newly proposed β velocity is more accurate and independent of any tuning coefficient while guaranteeing stability in presence of strong impacts. Furthermore, MUSCL reconstruction emerges as the optimal choice, as it significantly enhances the accuracy and stability of the scheme while making the scheme less dependent of the adopted β velocity formulations. However, it is nearly three times more computationally expensive.

REFERENCES

[1] Monaghan, J.J., 1997. SPH and Riemann solvers. *Journal of Computational Physics*, *136*(2), pp.298-307.

[2] Parshikov, A.N. and Medin, S.A., 2002. Smoothed particle hydrodynamics using interparticle contact algorithms. *Journal of computational physics*, *180*(1), pp.358-382.

[3] Zhang, C., Hu, X.Y. and Adams, N.A., 2017. A weakly compressible SPH method based on a low-dissipation Riemann solver. *Journal of Computational Physics*, *335*, pp.605-620.

[4] Rezavand, M., Zhang, C. and Hu, X., 2020. A weakly compressible SPH method for violent multi-phase flows with high density ratio. *Journal of Computational Physics*, *402*, p.109092.

[5] Chen, S.S., Li, J.P., Li, Z., Yuan, W. and Gao, Z.H., 2022. Anti-dissipation pressure correction under low Mach numbers for Godunov-type schemes. *Journal of Computational Physics*, *456*, p.111027.