Towards high-order simulation of compressible flows: insights from dispersion analysis

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I. Introduction

High-order methods were originally developed to simulate compressible flows with discontinuities. For such problems, high-order schemes allow minimizing numerical costs compared to low-order schemes. There are at least three ways to increase the order of SPH-approximation:

- the use of larger amount of summands of Taylor expansion in SPH-approximation;
- the use of ideas of WENO reconstruction (e.g. [1])
- the change of positive kernel to alternating one (for parabolic equations see [2], [3]).

The study shows some new results on the application of the third approach to the hyperbolic equations. This approach has a definite appeal since it allows improving the order of convergence with minimal changes to the SPH-code. Fig. 1 demonstrates the advantages of using a high-order kernel on the problem of Rihtmyer-Meshkov instability development. The bottom panel shows the result with a high-order kernel and the top panel for the classical kernel. One can see that small-scale vortices are visualized much better on the bottom panel, while the same number of particles and the same timestep is used. However, obtaining this figure with the classical SPH scheme with artificial viscosity and thermal conductivity (see details in [4]) requires fittig the used-defined parameters of viscosity and thermal conductivity, whilst its pre-defined values give unstable results. This reason motivated us for extra investigation of high-order SPH for compressible flows.

II. New method: motivation and idea

Classical interpolation formula of SPH is

$$F(x) \approx \sum_a F_a \frac{m_a W(x - x_a, H)}{\rho(x_a)}. \quad (1)$$

To find the first derivative of the quantity defined by the relation (1), the differentiation operation is applied to the smooth kernel:

$$\frac{\partial F(x)}{\partial x} \approx \sum_a \frac{F_a m_a}{\rho(x_a)} \frac{\partial W(x - x_a, H)}{\partial x}. \quad (2)$$

It turns out that using the kernel differentiation approach to approximate the second derivative leads to the scheme

$$\frac{\partial^2 F(x)}{\partial x^2} \approx \sum_a \frac{F_a m_a}{\rho(x_a)} \frac{\partial^2 W(x - x_a, H)}{\partial x^2}, \quad (3)$$

which has vital disadvantages – conditional approximation (error reduction can be achieved only by using a certain ratio between the number of neighbors in the smoothing radius and the number of smoothing radii in the wavelength) and tensile instability when using high-order alternating kernels. Mathematical justification of these features can be found, e.g., in [5].

It turned out that the alternative to the formula (3)

$$\frac{\partial^2 F(x_a)}{\partial x^2} = 2 \sum_b \frac{m_b}{\rho_b} \frac{F_a - F_b}{x_a - x_b} \frac{\partial W_{a,b}}{\partial x_a}, \quad (4)$$

successfully eliminates both disadvantages, i.e., it has unconditional approximation and is free from tensile instability when moving to high-order kernels [5]. At the same time, the use of (4) does not lead to an increase in computational costs compared to the basic idea (3).

The idea of the formula (4) is to calculate the first derivative of a function based on the finite-difference approach and then apply the kernel differentiation technique to find the derivative of the found value. Applying the same approach to constructing the first derivative gives

$$\frac{\partial F(x_a)}{\partial x} = \sum_{b \neq a} \frac{m_b}{\rho_b} \frac{F_a - F_b}{x_a - x_b} W_{a,b} \Big/ \left[1 - \frac{m_a}{\rho_a} W_{a,a}\right]. \quad (5)$$

The denominator of the right-hand side (5) becomes zero only when the particle has no neighbors.

III. New method vs classical method

Let us compare two methods of gradient approximation theoretically and in practice. Table I presents two schemes for SPH-approximation of transport equation

$$u_t + c u_x = 0, \quad (6)$$

where $u(x,t)$ – unknown function, $c > 0$ – constant parameter. The second column of the table presents approximate dispersion relations for these schemes. Mathematical and numerical analysis of these approximate dispersion relation allows to conclude

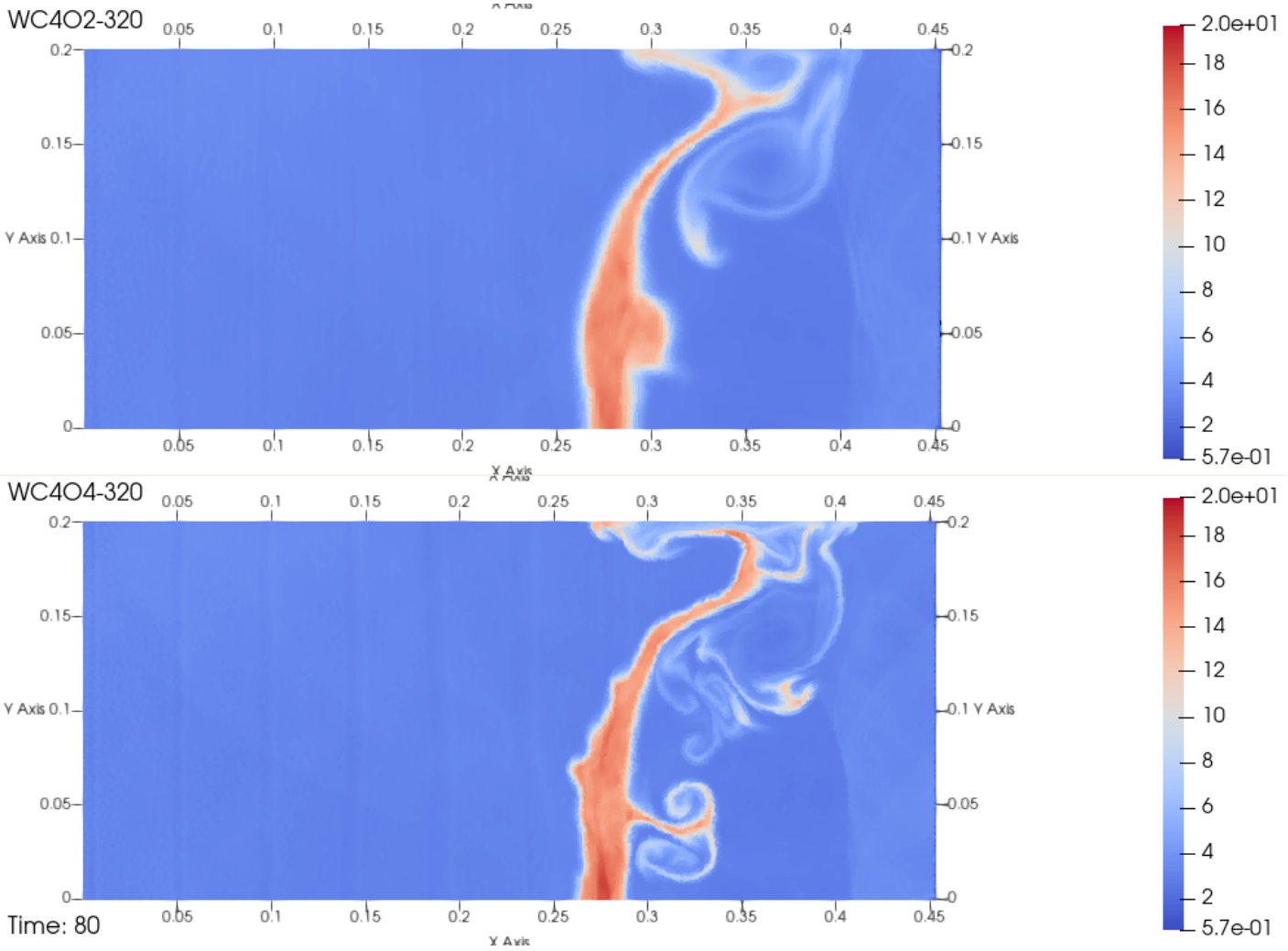

Fig. 1. Simulation of Rihtmyer-Meshkov instability development using high-order kernel and classical kernel with the same numerical setup.

that both schemes gives 4th order of approximation when C4O4 kernel is used (see details in [6]) This conclusion can be confirmed in practice as it is shown of Fig. 2. However, one can see that absolute error for new scheme is always small than for the classical one.

On the next step we are going to obtain dispesion relation for classical and new SPH-approximation of gas dynamics equations used for simulation of RM instability.

IV. Conclusions

This abstract proposes a new method for approximating first derivatives in SPH, based on the use of the idea of finite differences instead of kernel's differentiation. This method is an extension to first-order derivatives of the classical approach for calculating second-order derivatives, which is used in practice since 1985. The motivation for developing the new method was the problem of the stability of the classical method for approximating derivatives when using high-order alternating kernels.

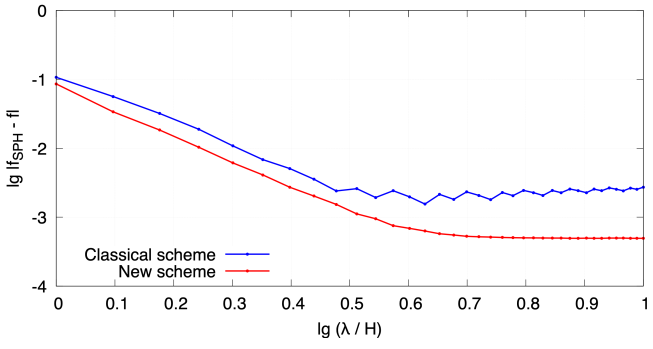

Fig. 2. The difference between numerical solution of (6) and analytical one as a function of λ/H. Here the analytical solution is $u(x,t) = u_0 \cos(k(x-ct))$, where $u_0 = 10^{-3}$, $k = 2\pi$, $\lambda = 2\pi/k$, and $c = 0.5$. The considered moment of time is $T = 10^{-1}$, the time step is $\tau = 10^{-3}$.

It is shown that with comparable computational costs, the new method for calculating derivatives gives the same order of

TABLE I
TABLE NAME

Scheme	**Approximate dispersion relation**
$\frac{du_a}{dt} = \sum_b \frac{m}{\rho_b}(u_a - c)u_b \frac{\partial W_{ab}}{\partial x_a}$	$\omega_{\text{ADR}} = ck\left(-\sum_{j=0}^{[1/\varphi]} \frac{\varphi K}{\pi} \sin\left(\frac{2\pi j\varphi}{K}\right)\tilde{W}_1(j\varphi)\right)$
$\frac{du_a}{dt} = \frac{\sum_{b\neq a} \frac{m}{\rho_b}(u_a - c)\frac{u_b - u_a}{x_b - x_a}W_{ab}}{1 - \frac{m}{\rho_a}W(0)}$	$\omega_{\text{ADR}} = ck\frac{K}{\pi(1-\varphi\tilde{W}(0))}\sum_{j=1}^{[1/\varphi]} \frac{\sin\left(\frac{2\pi j\varphi}{K}\right)\tilde{W}(j\varphi)}{j}$

[1] Shemes and approximate dispersion relations. Here $\tilde{W}_1$ and $\tilde{W}_2$ are first and second derivatives of dimensionless kernel $\tilde{W}$, $\varphi = \Delta x/H$, $K = \lambda/H$, where λ is wavelength, Δx is distance between neingbour particles.

approximation as the classical one. In addition, it is established that the actual error of the solution obtained by the new method is several times smaller than when using the classical method. This result is obtained theoretically by means of dispersion analysis and confirmed in practice. For this purpose, approximate dispersion relations for the new and classical methods of approximating the transport equation are derived.

V. Acknowledgement

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